# DIFFERENTIAL GEOMETRY FINAL EXAM

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The marks for each question is given in brackets after the question. The total is 60. This is a limited OPEN BOOK test – you may keep a copy of Shifrin's Differential Geometry book with you.

1. Let $\alpha$ be an arc length parametrized curve in $\mathbb{R}^3$. Suppose $\kappa\tau \neq 0$ at a point $P$. Of all the planes containing the tangent line to $\alpha$ at $P$, show that $\alpha$ lies locally on both sides only of the *osculating* plane. (6)

2. Show that a parametrization $x(u, v)$ of a surface is conformal if and only if $E = G$ and $F = 0$. (6)

3. Compute the second fundamental form $II_P$, the matrix of the Shape operator, $H$ and $K$ for the helicoid surface parametrized by

$$x(u, v) = (u\cos(v), u\sin(v), bv)$$

for some fixed constant $b$. (2,2,2,2)

4. Let $M$ be a surface such that, for every $P \in M$ the Shape operator, $S_P$ is some scalar multiple of the identity - that is

$$S_P(v) = k(P)v$$

for all $v \in T_P(M)$. Here $k(P)$ may depend on $P$. However, show that $k(P)$ is in fact a constant $k$ regardless of $P$ by computing $k_u$ and $k_v$. (6)

5. Calculate the Christoffel symbols of a surface of revolution $x(u, v) = (f(u)\cos(v), f(u)\sin(v), g(u))$ with $f'(u)^2 + g'(u)^2 = 1$. (8)

6. Prove there is no compact minimal surface in $M \subset \mathbb{R}^3$. (6)

7. Suppose $M$ is a surface with $F = 0$ and the $u$-curves are geodesics. Use the geodesic equations on page 77 to prove that $E$ is a function of $u$ only. (4)

8. Can their be a smooth, closed geodesic on a surface with $K = 0$ ? Prove why or why not. (6)

9. Consider the paraboloid $M$ parametrized by

$$x(u,v) = (u\cos(v), u\sin(v), u^2)$$

$0 \leq u, 0 \leq v \leq 2\pi$. Denote by $M_r$ the portion defined by $0 \leq u \leq r$.

(a) Compute the geodesic curvature of the boundary circle $u = r$ and compute $\int_{\partial M_r} \kappa_g ds$.. (3)

(b) Calculate $\chi(M_r)$. (2)

(c) Use the Gauss Bonnet Theorem to compute $\iint_{M_r} K dA$. (3)

(d) Compute the limit as $r \to \infty$. (2)